Lecture 4: A new shadowing result and application to Arnold diffusion

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- Justify all these facts requires a lot of technicalities.
- We will show a different mechanism that uses less knowledge of the inner dynamics in $\Lambda_{\theta,\varepsilon}$
- The basic idea is:
  If we know an orbit of the scattering map $x_{i+1} = S_\varepsilon(x_i)$, is it true that there is a real orbit of the Poincaré map $z_{i+1} = \mathcal{P}_{\theta,\varepsilon}^{k_i}(z_i)$ such that z_i is "close" to x_i?
  If this is true we just need to find orbits of the scattering map with increasing action.
  What we will see is that we have the following dichotomy:
  - The inner dynamics (which is the dynamics of $\mathcal{P}_{\theta,\varepsilon}$ restricted to $\Lambda_{\theta,\varepsilon}$) itself gives orbits which diffuse or
  - The outher dynamics (given by $\sigma_{\theta,\varepsilon}$) gives those orbits and we will find orbits of $\mathcal{P}_{\theta,\varepsilon}$ (given by $\sigma_{\theta,\varepsilon}$) which follows slosely them.
- We need a sahdowing lemma that does not need any invariant torus

A general Shadowing Lemma for NHIM's

Theorem 1 [Gidea, de la Llave, S.]

Given $f : M \to M$, is a C^r-map, $r \geq r_0$, $\Lambda \subseteq M$ NHIM, $\Gamma \subseteq M$ homoclinic channel. $\sigma = \sigma^\Gamma : \Omega^-(\Gamma) \to \Omega^+(\Gamma)$ is the scattering map associated to Γ. Assume that Λ and Γ are compact.
Then, for every $\delta > 0$ there exists $n^* \in \mathbb{N}$ and a family of functions $m_i^* : \mathbb{N}^{2i+1} \to \mathbb{N}$, $i \geq 0$, such that, for every pseudo-orbit $\{y_i\}_{i\geq 0}$ in Λ of the form

$$y_{i+1} = f^{m_i} \circ \sigma \circ f^{n_i}(y_i),$$

for all $i \geq 0$, with $n_i \geq n^*$ and $m_i \geq m_i^*(n_0, \dots, n_{i-1}, n_i, m_0, \dots, m_{i-1})$, there exists an orbit $\{z_i\}_{i\geq 0}$ of f in M such that, for all $i \geq 0$,

$$z_{i+1} = f^{m_i+n_i}(z_i), \quad \text{and} \quad d(z_i, y_i) < \delta.$$

One can use different scattering maps in the sequence!!
Related result: Gelfreich, Turaev Arnold Diffusion in a priori chaotic symplectic maps, Commun. Math. Phys., 2017

First tool: the λ-Lemma

$f : M \to M$, is a C^r-map, $r \geq r_0$, $\Lambda \subseteq M$ NHIM,
Let Δ be a 1-dimensional $\mathcal{C}^1$ submanifold transversely intersecting $W^s(\Lambda)$ at some point $p \in W^s(p_0)$ for some $p_0 \in \Lambda$.
Let $\Delta^k = f^k(\Delta)$, for $k \geq 1$. and set $p_0^k = f^k(p_0)$.
Then, there exist a neighborhoods U of Λ and $\forall \varepsilon > 0$, $\exists k_0$ and for $k \geq k_0$ there exists a $\mathcal{C}^1$-submanifold $\bar{\Delta}^k \subset \Delta^k$ such that

$$d_{C^1,U}(\bar{\Delta}^k, W^u(p_0^k) \cap U) < \varepsilon$$

Analogously, let Δ be a n_s-dimensional $\mathcal{C}^1$ submanifold transversely intersecting $W^u(\Lambda)$ at some point $p \in W^u(p_0)$, for some V.
Let $\Delta^k = f^{-k}(\Delta)$, for $k \geq 1$ and set $p_0^{-k} = f^{-k}(p_0)$. Then, there exist a neighborhood U of Λ, and $\forall \varepsilon > 0$, $\exists k_0$ and for $k \geq k_0$ there exists a $\mathcal{C}^1$-submanifold $\bar{\Delta}^{-k} \subset \Delta^{-k}$ such that

$$d_{C^1,U}(\bar{\Delta}^{-k}, W^s(p_0^{-k}) \cap U) < \varepsilon$$

Sabbagh, L. An inclination lemma for normally hyperbolic manifolds with an application to diffusion. Ergodic Theory Dynam. Systems 35 (2015),

A general Shadowing Lemma for NHIM's

- We have a pseudo-orbit: $y_{i+1} = f^{m_i} \circ \sigma \circ f^{n_i}(y_i)$
- The proof is based on the construction of a nested sequence of closed balls $B_{i+1} \subset B_i$ in a neighborhood of the first point of the pseudo-orbit y_0, such that taking $z_0 \in B_k = \bigcap_{0 \le i \le k} B_i$ one has that
  - $z_0 \in B_\delta(y_0)$
  - $z_{i+1} = f^{m_i + n_i}(z_i) \in B_\delta(y_{i+1})$ for $i = 0, 1 \ldots, k$, for any $k \in \mathcal{N}$.
  - Moreover, taking $z_0 \in B_\infty = \bigcap_{i \ge 0} B_i \ne \emptyset$, one has that: $z_{i+1} \in B_\delta(y_{i+1})$ for any $i \in \mathcal{N}$.
  - The argument will be done by induction.

Second tool: Poincaré recurrence

Definition

A point $x \in \Lambda$ is said to be recurrent for a map f on Λ, if for every open neighborhood $U \subseteq \Lambda$ of x, $f^k(x) \in U$ for some $k > 0$ large enough.

Theorem (Poincaré Recurrence Theorem)

Suppose that μ is a measure on Λ that is preserved by f, and $D \subset \Lambda$ is f-invariant with $\mu(D) < \infty$. Then μ-almost every point of D is recurrent.

Instead of recurrent points, in the arguments below we can use non-wandering points.

Proposition

Suppose that μ is a measure on Λ that is preserved by f, and $D \subset \Lambda$ is f-invariant with $\mu(D) < \infty$. Then every point $x \in D$ is non-wandering, that is, for every open neighborhood U of x in D, there exists $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$; moreover, n can be chosen arbitrarily large.

Shadowing Lemma for pseudo-orbits of the scattering map

Theorem 2 [Gidea, de la Llave, S.]

$f : M \to M$ smooth map, $\Lambda \subseteq M$ is a NHIM, $\Gamma \subseteq M$ homoclinic channel and σ is the scattering map associated to Γ.

f preserves a measure μ absolutely continuous with respect to the Lebesgue measure on Λ,

σ sends positive measure sets to positive measure sets.

Let $\{x_i\}_{i=0,\dots,n}$ be a finite pseudo-orbit of the scattering map in Λ, i.e., $x_{i+1} = \sigma(x_i)$, $i = 0, \dots, n-1$, $n \geq 1$, that is contained in some open set $\mathcal{U} \subseteq \Lambda$ with almost every point of $\mathcal{U}$ recurrent for $f_{|\Lambda}$. (The points $\{x_i\}_{i=0,\dots,n}$ do not have to be themselves recurrent.)

Then, for every $\delta > 0$ there exists an orbit $\{z_i\}_{i=0,\dots,n}$ of f in M, with $z_{i+1} = f^{k_i}(z_i)$ for some $k_i > 0$, such that $d(z_i, x_i) < \delta$ for all $i = 0, \dots, n$.

Shadowing Lemma for pseudo-orbits of the scattering map: Proof

- Choose a small open disk B_0 of x_0 in Λ, with $B_0 \subseteq \mathcal{U}$ such that $B_i := \sigma^i(B_0) \subseteq \mathcal{U}$, and $\mathrm{diam}(B_i) \leq \delta/2$, for all $i = 0, \ldots, n$.
- For the given pseudo-orbit $\{x_i\}$ of σ, with $x_{i+1} = \sigma(x_i)$, we have that $x_i \in B_i$ for all i.
- We will use Poincaré recurrence to produce a new pseudo-orbit $\{y_i\}$, with $y_{i+1} = f^{m_i} \circ \sigma \circ f^{n_i}(y_i)$, where m_i, n_i are as in previous theorem, such that $y_i \in B_i$ for all i, and hence $d(y_i, x_i) \leq \delta/2$.
- The shadowing theorem will provide us with a true orbit $\{z_i\}$ with $z_{i+1} = f^{m_i+n_i}(z_i)$, such that $d(z_i, y_i) \leq \delta/2$, hence $d(z_i, x_i) < \delta$.

Theorem 3 [Gidea, de la Llave, S.] A Perturbative result

Given H_ε. Assume for all $0 < \varepsilon < \varepsilon_0$ there exist

- NHIM Λ_ε
- Homoclinic channel Γ_ε and corresponding scattering map $\sigma_\varepsilon = \mathrm{Id} + \varepsilon J\nabla S + O(\varepsilon^2)$
- Suppose that $J\nabla S(x_0) \neq 0$ at some point $x_0 \in \Lambda$. Let $\tilde{\gamma} : [0,1] \to \Lambda_0$ be an integral curve through x_0 for the vector field $\dot{x} = J\nabla S(x)$.
- Suppose that there exists a neighborhood $\mathcal{U}$ of $\tilde{\gamma}([0,1])$ in Λ_ε such that a.e. point in $\mathcal{U}$ is recurrent for $F_{\varepsilon|\Lambda_0}$.

Then for every $\delta > 0$, there exists an orbit $\{z_i\}_{i=0,\dots,n}$ of F_ε in M, with $n = O(\mu(\varepsilon)^{-1})$, such that for all $i = 0, \dots, n-1$,

$$z_{i+1} = F_\varepsilon^{k_i}(z_i), \quad \text{for some } k_i > 0, \text{ and}$$

$$d(z_i, \gamma(t_i)) < \delta + K\varepsilon, \ \text{for } t_i = i \cdot \varepsilon$$

where $0 = t_0 < t_1 < \dots < t_n \leq 1$.

Proof of Theorem 3

The main idea is that the scattering map is given by $\sigma_\varepsilon = \mathrm{Id} + \varepsilon J\nabla S + O(\varepsilon^2)$ therefore, its orbits are close to the orbits obtained by applying the Euler method of step ε to the vector field

$$\dot{x} = J\nabla S(x)$$

Therefore, one can find an orbit $x_{i+1} = s_\varepsilon(x_i)$ such that

$$x_0 = \gamma(0), \quad x_{i+1} = s_\varepsilon(x_i) \in \mathcal{U} \subset \Lambda,$$

and

$$d(\gamma(t_i), x_i) < K\varepsilon, \quad i = 0, \dots, n, \; n = O(1/\varepsilon)$$

then we apply Theorem 2 to obtain an orbit $z_{i+1} = F_\varepsilon^{k_i}(z_i)$ in M, for some $k_i > 0$, s.t. $d(z_i, x_i) < \delta$ for all $i = 0, \dots, n$

A general diffusion result

Corollary [Gidea, de la Llave, S.]

Given $H_\varepsilon = H_0 + \varepsilon H_1$. Assume for all $0 < \varepsilon < \varepsilon_0$ there exist

- NHIM $\Lambda_\varepsilon = k_\varepsilon(\Lambda_0)$
- Homoclinic channel Γ_ε and corresponding scattering map $s_\varepsilon = \mathrm{Id} + \varepsilon J\nabla S + O(\varepsilon^2)$, where $s_\varepsilon = k_\varepsilon^{-1} \circ \sigma_\varepsilon \circ k_\varepsilon$
- $\Lambda_0 \subseteq \mathbb{R}^d \times \mathbb{T}^d \ni (I, \varphi)$

If $J\nabla S(I,\varphi)$ is transverse to some level set $\{I = I_*\}$ of I, then $\exists \varepsilon_1 < \varepsilon_0$, $\exists C > 0$, s.t. $\forall \varepsilon < \varepsilon_1$ $\exists x(t)$ with

$$\|I(x(T)) - I(x(0))\| > C, \ \text{for some } \ T > 0.$$

- Remark:
  - There are no requirements on the inner dynamics, except of being conservative

Proof of the Corollary

- Given $J\nabla S(I,\varphi)$ transverse to $\{I = I_0\}$
  $\Rightarrow$ $J\nabla S(I,\varphi)$ transverse to $\{I = I_*\}$ with $\|I_* - I_0\| < \delta$, for some $\delta > 0$ independent of ε
  $\Rightarrow$ there is a strip $\mathcal{S}$ of φ-size $O(1)$ consisting of trajectories of the Hamiltonian system $\dot{x} = J\nabla S(x)$ along which I changes $O(1)$.
  $\Rightarrow$ there are orbits of the map s_ε along which I changes $O(1)$
- We have two possibilities
  - There is a bounded domain through the inner dynamics, then we have Poincaré recurrence and Theorem 3 applies
  - There is diffusion using only the inner dynamics

Application

Diffusion in an a priori unstable system

$$H_\varepsilon(p,q,I,\varphi,t) = \underbrace{h_0(I) + \sum_{i=1}^{n} \pm \left(\frac{1}{2}p_i^2 + V_i(q_i)\right)}_{H_0} + \varepsilon H_1(p,q,I,\varphi,t;\varepsilon),$$

$(p,q,I,\varphi,t) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{T}^1$

Theorem 4 [Gidea, de la Llave, S.]

Under the earlier assumptions,
there exists $\varepsilon_0 > 0$, and $C > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a trajectory $x(t)$ such that

$$\|I(x(T)) - I(x(0))\| > C \text{ for some } T > 0.$$

- We make no asumptions on the dynamics of h_0. No need of KAM tori, Aubry Mather sets etc, do not require any property on $\partial^2 h_0/\partial I^2 \neq 0$
- No convexity of the unperturbed Hamiltonian; the argument works even if $\partial^2 h_0/\partial I^2$ degenerate or non-positive definite (e.g., non-twist maps)
- We allow strong resonances etc.
- Any dimension.
- Works for perturbations in an open and dense set satisfying explicit non-degeneracy conditions

Proofs

- Proof of Theorem 4:
- Penduli $\rightsquigarrow$ homoclinic orbit $(p_i^0(\sigma), q_i^0(\sigma))$ to $(0,0)$
- Let
$$L(\tau, I, \varphi, s) = -\int_{-\infty}^{\infty} \Big[H_1(p^0(\tau+\sigma), q^0(\tau+\sigma), I, \varphi+\omega(I)\sigma, s+\sigma; 0) \\ -H_1(0,0,I,\varphi+\omega(I)\sigma, s+\sigma; 0)\Big]\, dt$$
- For generic H_1, the equation $\frac{\partial}{\partial\tau} L(\tau, I, \varphi, s) = 0$ has a non degenerate solution $\tau = \tau^*(I, \varphi, s)$
- Define $\mathcal{L}(I, \varphi, s) = L(\tau^*(I, \varphi, s), I, \varphi, s)$ and $\mathcal{L}^*(I, \theta) = \mathcal{L}(I, \theta, 0)$
- $s_\varepsilon(I, \varphi) = \mathrm{Id}(I, \varphi) + \varepsilon J \nabla \mathcal{L}^*(I, \varphi - \omega(I)s) + O(\varepsilon^2)$
- For generic H_1, $\nabla \mathcal{L}^*$ is transverse to some level set $\{I = I_0\}$
- Apply Theorem 3 and Corollary.

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